

## ORIGINAL RESEARCH

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# The generalized $\chi^2$ sequence spaces over $p$ - metric spaces defined by Musielak

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In this paper, we introduce generalized  $\chi^2$  sequence spaces over  $p$ - metric spaces defined by Musielak function  $f = (f_{mn})$  and study some topological properties.

**Keywords:** Analytic sequence; Double sequences;  $\chi^2$  space; Difference sequence space; Musielak-modulus function;  $p$ - metric space; Duals

**MSC:** 40A05; 40C05; 40D05

**Introduction**

Throughout this paper,  $w$ ,  $\chi$ , and  $\Lambda$  denote the classes of all, gai, and analytic scalar valued single sequences, respectively. We write  $w^2$  for the set of all complex sequences  $(x_{mn})$ , where  $m, n \in \mathbb{N}$ , the set of positive integers. Then,  $w^2$  is a linear space under the coordinatewise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [1]. Later on, they were investigated by Hardy [2], Moricz [3], Moricz and Rhoades [4], Basarir and Solankan [5], Tripathy [6], Turkmenoglu [7], and many others. We procure the following sets of double sequences:

$$\mathcal{M}_u(t) := \{(x_{mn}) \in w^2 : \sup_{m,n \in \mathbb{N}} |x_{mn}|^{t_{mn}} < \infty\},$$

$$\mathcal{C}_p(t) := \{(x_{mn}) \in w^2 : p - \lim_{m,n \rightarrow \infty} |x_{mn}|^{t_{mn}} = 1 \text{ for some } \in \mathbb{C}\},$$

$$\mathcal{C}_{0p}(t) := \{(x_{mn}) \in w^2 : p - \lim_{m,n \rightarrow \infty} |x_{mn}|^{t_{mn}} = 1\},$$

$$\mathcal{L}_u(t) := \{(x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \cap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \cap \mathcal{M}_u(t),$$

where  $t = (t_{mn})$  is the sequence of strictly positive reals  $t_{mn}$  for all  $m, n \in \mathbb{N}$  and  $p - \lim_{m,n \rightarrow \infty}$  denotes the limit in the Pringsheim's sense. In the case where  $t_{mn} =$

1 for all  $m, n \in \mathbb{N}$ ,  $\mathcal{M}_u(t)$ ,  $\mathcal{C}_p(t)$ ,  $\mathcal{C}_{0p}(t)$ ,  $\mathcal{L}_u(t)$ ,  $\mathcal{C}_{bp}(t)$ , and  $\mathcal{C}_{0bp}(t)$  reduce to the sets  $\mathcal{M}_u$ ,  $\mathcal{C}_p$ ,  $\mathcal{C}_{0p}$ ,  $\mathcal{L}_u$ ,  $\mathcal{C}_{bp}$ , and  $\mathcal{C}_{0bp}$ , respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [8,9] have proved that  $\mathcal{M}_u(t)$  and  $\mathcal{C}_p(t)$ ,  $\mathcal{C}_{bp}(t)$  are complete paranormed spaces of double sequences and gave the  $\alpha$ -,  $\beta$ -,  $\gamma$ -duals of the spaces  $\mathcal{M}_u(t)$  and  $\mathcal{C}_{bp}(t)$ . Quite recently, in her PhD thesis, Zeltser [10] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [11], and Tripathy [6] have independently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Altay and Başar [12] have defined the spaces  $\mathcal{BS}$ ,  $\mathcal{BS}(t)$ ,  $\mathcal{CS}_p$ ,  $\mathcal{CS}_{bp}$ ,  $\mathcal{CS}_r$ , and  $\mathcal{BV}$  of double sequences consisting of all double series whose sequence of partial sums is in the spaces  $\mathcal{M}_u$ ,  $\mathcal{M}_u(t)$ ,  $\mathcal{C}_p$ ,  $\mathcal{C}_{bp}$ ,  $\mathcal{C}_r$ , and  $\mathcal{L}_u$ , respectively, and also examined some properties of those sequence spaces and determined the  $\alpha$ -duals of the spaces  $\mathcal{BS}$ ,  $\mathcal{BV}$ ,  $\mathcal{CS}_{bp}$ , and the  $\beta(\vartheta)$ -duals of the spaces  $\mathcal{CS}_{bp}$  and  $\mathcal{CS}_r$  of double series. Basar and Sever [13] have introduced the Banach space  $\mathcal{L}_q$  of double sequences corresponding to the well-known space  $\ell_q$  of single sequences and examined some properties of the space  $\mathcal{L}_q$ . Quite recently, Subramanian and Misra [14] have studied the space  $\chi_M^2(p, q, u)$  of double sequences and gave some inclusion relations.

The class of sequences which is strongly Cesàro summable with respect to a modulus was introduced

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by Maddox [15] as an extension of the definition of strongly Cesàro summable sequences. Connor [16] further extended this definition to a definition of strong  $A$ -summability with respect to a modulus, where  $A = (a_{n,k})$  is a non-negative regular matrix, and established some connections between strong  $A$ -summability, strong  $A$ -summability with respect to a modulus, and  $A$ -statistical convergence. In [17], the notion of convergence of double sequences was presented by Pringsheim. Also, in [18,19], and [20], the four-dimensional matrix transformation  $(Ax)_{k,\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$  was studied extensively by Hamilton.

We need the following inequality in the sequel of the paper. For  $a, b, \geq 0$  and  $0 < p < 1$ , we have

$$(a + b)^p \leq a^p + b^p. \quad (1)$$

The double series  $\sum_{m,n=1}^{\infty} x_{mn}$  is called convergent if and only if the double sequence  $(s_{mn})$  is convergent, where  $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij}$  ( $m, n \in \mathbb{N}$ ). A sequence  $x = (x_{mn})$  is said to be double analytic if  $\sup_{mn} |x_{mn}|^{1/m+n} < \infty$ . The vector space of all double analytic sequences will be denoted by  $\Lambda^2$ . A sequence  $x = (x_{mn})$  is called double gai sequence if  $((m+n)! |x_{mn}|)^{1/m+n} \rightarrow 0$  as  $m, n \rightarrow \infty$ . The double gai sequences will be denoted by  $\chi^2$ . Let  $\phi = \{\text{all finite sequences}\}$ .

Consider a double sequence  $x = (x_{ij})$ . The  $(m, n)^{\text{th}}$  section  $x^{[m,n]}$  of the sequence is defined by  $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \mathfrak{S}_{ij}$  for all  $m, n \in \mathbb{N}$ , where  $\mathfrak{S}_{ij}$  denotes the double sequence whose only non-zero term is a  $\frac{1}{(i+j)!}$  in the  $(i, j)^{\text{th}}$  place for each  $i, j \in \mathbb{N}$ .

A Fréchet coordinate space (FK-space or a metric space)  $X$  is said to have an AK property if  $(\mathfrak{S}_{mn})$  is a Schauder basis for  $X$ , or equivalently  $x^{[m,n]} \rightarrow x$ . An FDK-space is a double sequence space endowed with a complete metrizable space, locally convex topology under which the coordinate mappings  $x = (x_k) \rightarrow (x_{mn})(m, n \in \mathbb{N})$  are also continuous.

Let  $M$  and  $\Phi$  be mutually complementary modulus functions. Then, we have

- (1) For all  $u, y \geq 0$ ,

$$uy \leq M(u) + \Phi(y), \text{ (Young's inequality; see [21])}. \quad (2)$$

- (2) For all  $u \geq 0$ ,

$$u\eta(u) = M(u) + \Phi(\eta(u)). \quad (3)$$

- (3) For all  $u \geq 0$  and  $0 < \lambda < 1$ ,

$$M(\lambda u) \leq \lambda M(u). \quad (4)$$

Lindenstrauss and Tzafriri [22] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

The space  $\ell_M$  with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

becomes a Banach space which is called an Orlicz sequence space. For  $M(t) = t^p$  ( $1 \leq p < \infty$ ), the spaces  $\ell_M$  coincide with the classical sequence space  $\ell_p$ .

A sequence  $f = (f_{mn})$  of modulus function is called a Musielak-modulus function. A sequence  $g = (g_{mn})$  defined by

$$g_{mn}(v) = \sup \{ |v| u - (f_{mn})(u) : u \geq 0 \}, m, n = 1, 2, \dots$$

is called the complementary function of a Musielak-modulus function  $f$ . For a given Musielak modulus function  $f$ , the Musielak-modulus sequence space  $t_f$  and its subspace  $h_f$  are defined, respectively, as follows:

$$t_f = \{ x \in w^2 : I_f(|x_{mn}|)^{1/m+n} \rightarrow 0 \text{ as } m, n \rightarrow \infty \}$$

and

$$h_f = \{ x \in w^2 : I_f(|x_{mn}|)^{1/m+n} \rightarrow 0 \text{ as } m, n \rightarrow \infty \},$$

where  $I_f$  is a convex modular defined by

$$I_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn}(|x_{mn}|)^{1/m+n}, x = (x_{mn}) \in t_f.$$

We consider that  $t_f$  is equipped with the Luxemburg metric

$$d(x, y) = \sup_{mn} \left\{ \inf \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn} \times \left( \frac{|x_{mn}|^{1/m+n}}{mn} \right) \right) \leq 1 \right\}.$$

If  $X$  is a sequence space, we give the following definitions:

- (1)  $X'$  = the continuous dual of  $X$ ;
- (2)  $X^\alpha = \{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn} x_{mn}| < \infty, \text{ for each } x \in X \}$ ;
- (3)  $X^\beta = \{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn} x_{mn} \text{ is convergent, for each } x \in X \}$ ;
- (4)  $X^\gamma = \{ a = (a_{mn}) : \sup_{mn} \geq 1 \left| \sum_{m,n=1}^{M,N} a_{mn} x_{mn} \right| < \infty, \text{ for each } x \in X \}$ ;
- (5) let  $X$  be an FK-space  $\supset \phi$ , then  $X^f = \{ f(\mathfrak{S}_{mn}) : f \in X' \}$ ;
- (6)  $X^\delta = \{ a = (a_{mn}) : \sup_{mn} |a_{mn} x_{mn}|^{1/m+n} < \infty, \text{ for each } x \in X \}$ ,

where  $X^\alpha$ ,  $X^\beta$ , and  $X^\gamma$  are called  $\alpha$ - (or Köthe-Toeplitz) dual of  $X$ ,  $\beta$ - (or generalized Köthe-Toeplitz) dual of  $X$ ,  $\gamma$ - dual of  $X$ , and  $\delta$ - dual of  $X$ , respectively.  $X^\alpha$  is defined

by Kantham and Gupta [21]. It is clear that  $X^\alpha \subset X^\beta$  and  $X^\alpha \subset X^\gamma$ , but  $X^\beta \subset X^\gamma$  does not hold since the sequence of partial sums of a double convergent series needs not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [23] as follows:

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for  $Z = c, c_0$  and  $\ell_\infty$ , where  $\Delta x_k = x_k - x_{k+1}$  for all  $k \in \mathbb{N}$ .

Here,  $c$ ,  $c_0$ , and  $\ell_\infty$  denote the classes of convergent, null, and bounded scalar valued single sequences, respectively. The difference sequence space  $bv_p$  of the classical space  $\ell_p$  is introduced and studied in the case  $1 \leq p \leq \infty$  and in the case  $0 < p < 1$  by Altay and Bařar in [12]. The spaces  $c(\Delta)$ ,  $c_0(\Delta)$ ,  $\ell_\infty(\Delta)$ , and  $bv_p$  are Banach spaces normed by

$$\|x\| = |x_1| + \sup_{k \geq 1} |\Delta x_k| \text{ and } \|x\|_{bv_p} = \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}, (1 \leq p < \infty).$$

Later on, the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\},$$

where  $Z = \Delta^2, \chi^2$  and  $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$  for all  $m, n \in \mathbb{N}$ .

### Definition and preliminaries

Let  $n \in \mathbb{N}$  and  $X$  be a real vector space of dimension  $w$ , where  $n \leq w$ . A real valued function  $d_p(x_1, \dots, x_n) = \|(d_1(x_1), \dots, d_n(x_n))\|_p$  on  $X$  satisfying the following four conditions:

- (1)  $\|(d_1(x_1), \dots, d_n(x_n))\|_p = 0$  if and only if  $d_1(x_1), \dots, d_n(x_n)$  are linearly dependent,
- (2)  $\|(d_1(x_1), \dots, d_n(x_n))\|_p$  is invariant under permutation,
- (3)  $\|(\alpha d_1(x_1), \dots, \alpha d_n(x_n))\|_p = |\alpha| \|(d_1(x_1), \dots, d_n(x_n))\|_p, \alpha \in \mathbb{R}$
- (4)  $d_p((x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)) = (d_X(x_1, x_2, \dots, x_n)^p + d_Y(y_1, y_2, \dots, y_n)^p)^{1/p}$  for  $1 \leq p < \infty$ ; (or)
- (5)  $d((x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)) := \sup \{d_X(x_1, x_2, \dots, x_n), d_Y(y_1, y_2, \dots, y_n)\}$ , for  $x_1, x_2, \dots, x_n \in X, y_1, y_2, \dots, y_n \in Y$  which is called the  $p$  product metric of the Cartesian product of  $n$  metric spaces is the  $p$  norm of the  $n$ -vector of the norms of the  $n$  subspaces.

A trivial example of the  $p$  product metric of the  $n$  metric space is the  $p$  norm space which is  $X = \mathbb{R}$  equipped with the following Euclidean metric in the product space:

$$\|(d_1(x_1), \dots, d_n(x_n))\|_E = \sup (|\det(d_{mn}(x_{mn}))|) = \sup \left( \begin{vmatrix} d_{11}(x_{11}) & d_{12}(x_{12}) & \dots & d_{1n}(x_{1n}) \\ d_{21}(x_{21}) & d_{22}(x_{22}) & \dots & d_{2n}(x_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1}(x_{n1}) & d_{n2}(x_{n2}) & \dots & d_{nn}(x_{nn}) \end{vmatrix} \right),$$

where  $x_i = (x_{i1}, \dots, x_{in}) \in \mathbb{R}^n$  for each  $i = 1, 2, \dots, n$ .

If every Cauchy sequence in  $X$  converges to some  $L \in X$ , then  $X$  is said to be complete with respect to the  $p$ - metric. Any complete  $p$ - metric space is said to be  $p$ - Banach metric space.

Let  $X$  be a linear metric space. A function  $w : X \rightarrow \mathbb{R}$  is called paranorm if

- (1)  $w(x) \geq 0$  for all  $x \in X$ ;
- (2)  $w(-x) = w(x)$  for all  $x \in X$ ,
- (3)  $w(x + y) \leq w(x) + w(y)$  for all  $x, y \in X$ ;
- (4) If  $(\sigma_{mn})$  is a sequence of scalars with  $\sigma_{mn} \rightarrow \sigma$  as  $m, n \rightarrow \infty$ , and  $(x_{mn})$  is a sequence of vectors with  $w(x_{mn} - x) \rightarrow 0$  as  $m, n \rightarrow \infty$ , then  $w(\sigma_{mn}x_{mn} - \sigma x) \rightarrow 0$  as  $m, n \rightarrow \infty$ .

A paranorm  $w$  for which  $w(x) = 0$  implies  $x = 0$  is called a total paranorm, and the pair  $(X, w)$  is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [24], Theorem 10.4.2, p.183).

The notion of  $\lambda$ - double gai and double analytic sequences is as follows: Let  $\lambda = (\lambda_{mn})_{m,n=0}^{\infty}$  be a strictly increasing sequence of positive real numbers tending to infinity, that is,

$$0 < \lambda_0 < \lambda_1 < \dots \text{ and } \lambda_{mn} \rightarrow \infty \text{ as } m, n \rightarrow \infty$$

and that a sequence  $x = (x_{mn}) \in w^2$  is  $\lambda$ - convergent to 0, called a the  $\lambda$ - limit of  $x$ , if  $\mu_{mn}(x) \rightarrow 0$  as  $m, n \rightarrow \infty$ , where

$$\mu_{mn}(x) = \frac{1}{\varphi_{rs}} \sum_{m \in \sigma, \sigma \in P_{rs}} \sum_{n \in \sigma, \sigma \in P_{rs}} (\lambda_{m,n} - \lambda_{m,n+1} - \lambda_{m+1,n} + \lambda_{m+1,n+1}) |x_{mn}|^{1/m+n}.$$

The sequence  $x = (x_{mn}) \in w^2$  is  $\lambda$ - double analytic if  $\sup_{uv} |\mu_{mn}(x)| < \infty$ . If  $\lim_{mn} x_{mn} = 0$  in the ordinary sense of convergence, then

$$\lim_{mn} \left( \frac{1}{\varphi_{rs}} \sum_{m \in \sigma, \sigma \in P_{rs}} \sum_{n \in \sigma, \sigma \in P_{rs}} (\lambda_{m,n} - \lambda_{m,n+1} - \lambda_{m+1,n} + \lambda_{m+1,n+1}) ((m+n)! |x_{mn} - 0|)^{1/m+n} \right) = 0.$$

This implies that it yields  $\lim_{uv} \mu_{mn}(x) = 0$ , and hence,  $x = (x_{mn}) \in w^2$  is  $\lambda$ - convergent to 0. Let  $f = (f_{mn})$  be a Musielak-modulus function,  $(X, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p)$  be a  $p$ -metric space,

and  $q = (q_{mn})$  be double analytic sequence of strictly positive real numbers. By  $w^2(p - X)$ , we denote the space of all sequences as  $(X, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p)$ . The following inequality will be used throughout the paper. If  $0 \leq q_{mn} \leq \sup q_{mn} = H, K = \max(1, 2^{H-1})$ , then

$$|a_{mn} + b_{mn}|^{q_{mn}} \leq K \{ |a_{mn}|^{q_{mn}} + |b_{mn}|^{q_{mn}} \} \quad (5)$$

for all  $m, n$  and  $a_{mn}, b_{mn} \in \mathbb{C}$ . Also,  $|a|^{q_{mn}} \leq \max(1, |a|^H)$  for all  $a \in \mathbb{C}$ .

In the present paper, we define the following sequence spaces:

$$\begin{aligned} & [\chi_{f\mu}^{2q}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi] \\ &= \lim_{mn} \left\{ \left[ f_{mn} \left( \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, \right. \right. \right. \\ & \quad \left. \left. \left. d(x_{n-1}))\|_p \right) \right]^{q_{mn}} = 0 \right\}, \\ & [\Lambda_{f\mu}^{2q}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi] \\ &= \sup_{mn} \left\{ \left[ f_{mn} \left( \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, \right. \right. \right. \right. \\ & \quad \left. \left. \left. d(x_{n-1}))\|_p \right) \right]^{q_{mn}} < \infty \right\}. \end{aligned}$$

If we take  $f_{mn}(x) = x$ , we get

$$\begin{aligned} & [\chi_{\mu}^{2q}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi] \\ &= \lim_{mn} \left\{ \left[ \left( \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, \right. \right. \right. \right. \\ & \quad \left. \left. \left. d(x_{n-1}))\|_p \right) \right]^{q_{mn}} = 0 \right\}, \\ & [\Lambda_{\mu}^{2q}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi] \\ &= \sup_{mn} \left\{ \left[ \left( \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, \right. \right. \right. \right. \\ & \quad \left. \left. \left. d(x_{n-1}))\|_p \right) \right]^{q_{mn}} < \infty \right\}. \end{aligned}$$

If we take  $q = (q_{mn}) = 1$

$$\begin{aligned} & [\chi_{f\mu}^2, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi] \\ &= \lim_{mn} \left\{ \left[ f_{mn} \left( \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, \right. \right. \right. \right. \\ & \quad \left. \left. \left. d(x_{n-1}))\|_p \right) \right] = 0 \right\}, \\ & [\Lambda_{f\mu}^2, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi] \\ &= \sup_{mn} \left\{ \left[ f_{mn} \left( \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, \right. \right. \right. \right. \\ & \quad \left. \left. \left. d(x_{n-1}))\|_p \right) \right] < \infty \right\}. \end{aligned}$$

In the present paper, we plan to study some topological properties and inclusion relation between the above defined sequence spaces,  $[\chi_{f\mu}^{2q}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi]$  and  $[\Lambda_{f\mu}^{2q}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi]$ , which we shall discuss in this paper.

## Main results

**Theorem 1.** Let  $f = (f_{mn})$  be a Musielak-modulus function and  $q = (q_{mn})$  be a double analytic sequence of strictly positive real numbers; the sequence spaces  $[\chi_{f\mu}^{2q}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi]$  and  $[\Lambda_{f\mu}^{2q}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi]$  are linear spaces.

*Proof.* It is routine verification. Therefore, the proof is omitted.  $\square$

**Theorem 2.** Let  $f = (f_{mn})$  be a Musielak-modulus function and  $q = (q_{mn})$  be a double analytic sequence of strictly positive real numbers; the sequence space  $[\chi_{f\mu}^{2q}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi]$  is a paranormed space with respect to the paranorm defined by

$$g(x) = \inf \left\{ \left( \left[ f_{mn} \left( \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right) \right]^{q_{mn}} \right)^{1/H} \leq 1 \right\},$$

where  $H = \max(1, \sup_{mn} q_{mn} < \infty)$ .

*Proof.* Clearly,  $g(x) \geq 0$  for  $x = (x_{mn}) \in [\chi_{f\mu}^{2q}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{V_2}]$ . Since  $f_{mn}(0) = 0$ , we get  $g(0) = 0$ .

Conversely, suppose that  $g(x) = 0$ , then

$$\inf \left\{ \left( \left[ f_{mn} \left( \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right) \right]^{q_{mn}} \right)^{1/H} \leq 1 = 0.$$

Suppose that  $\mu_{mn}(x) \neq 0$  for each  $m, n \in \mathbb{N}$ . Then,  $\|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \rightarrow \infty$ . It follows that  $\left( \left[ f_{mn} \left( \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{V_2} \right) \right]^{q_{mn}} \right)^{1/H} \rightarrow \infty$  which is a contradiction. Therefore,  $\mu_{mn}(x) = 0$ . Let

$$\begin{aligned} & \left( \left[ f_{mn} \left( \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right) \right]^{q_{mn}} \right)^{1/H} \leq 1 \\ & \text{and} \\ & \left( \left[ f_{mn} \left( \|\mu_{mn}(y), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right) \right]^{q_{mn}} \right)^{1/H} \leq 1. \end{aligned}$$

Then, by using Minkowski's inequality, we have

$$\begin{aligned} & \left( \left[ f_{mn} \left( \left\| \mu_{mn}(x+y), (d(x_1), d(x_2), \dots, d(x_{n-1})) \right\|_p^\varphi \right) \right]^{q_{mn}} \right)^{1/H} \\ & \leq \left( \left[ f_{mn} \left( \left\| \mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1})) \right\|_p^\varphi \right) \right]^{q_{mn}} \right)^{1/H} \\ & \quad + \left( \left[ f_{mn} \left( \left\| \mu_{mn}(y), (d(x_1), d(x_2), \dots, d(x_{n-1})) \right\|_p^\varphi \right) \right]^{q_{mn}} \right)^{1/H}. \end{aligned}$$

So, we have

$$\begin{aligned} g(x+y) &= \inf \left\{ \left( \left[ f_{mn} \left( \left\| \mu_{mn}(x+y), (d(x_1), d(x_2), \dots, \right. \right. \right. \right. \right. \\ & \quad \left. \left. \left. \left. \left. d(x_{n-1})) \right\|_p^\varphi \right) \right]^{q_{mn}} \right)^{1/H} \leq 1 \right\} \\ & \leq \inf \left\{ \left( \left[ f_{mn} \left( \left\| \mu_{mn}(x), (d(x_1), d(x_2), \dots, \right. \right. \right. \right. \right. \right. \\ & \quad \left. \left. \left. \left. \left. d(x_{n-1})) \right\|_p^\varphi \right) \right]^{q_{mn}} \right)^{1/H} \leq 1 \right\} \\ & \quad + \inf \left\{ \left( \left[ f_{mn} \left( \left\| \mu_{mn}(y), (d(x_1), d(x_2), \dots, \right. \right. \right. \right. \right. \right. \\ & \quad \left. \left. \left. \left. \left. d(x_{n-1})) \right\|_p^\varphi \right) \right]^{q_{mn}} \right)^{1/H} \leq 1 \right\}. \end{aligned}$$

Therefore,

$$g(x+y) \leq g(x) + g(y).$$

Finally, to prove that the scalar multiplication is continuous, let  $\lambda$  be any complex number. By definition,

$$g(\lambda x) = \inf \left\{ \left( \left[ f_{mn} \left( \left\| \mu_{mn}(\lambda x), (d(x_1), d(x_2), \dots, \right. \right. \right. \right. \right. \right. \right. \\ \left. \left. \left. \left. \left. d(x_{n-1})) \right\|_p^\varphi \right) \right]^{q_{mn}} \right)^{1/H} \leq 1 \right\}.$$

Then,

$$g(\lambda x) = \inf \left\{ (|\lambda| t)^{q_{mn}/H} : \left( \left[ f_{mn} \left( \left\| \mu_{mn}(x), (d(x_1), d(x_2), \dots, \right. \right. \right. \right. \right. \right. \right. \\ \left. \left. \left. \left. \left. d(x_{n-1})) \right\|_p^\varphi \right) \right]^{q_{mn}} \right)^{1/H} \leq 1 \right\},$$

where  $t = \frac{1}{|\lambda|}$ . Since  $|\lambda|^{q_{mn}} \leq \max(1, |\lambda|^{supp_{mn}})$ , we have

$$\begin{aligned} g(\lambda x) &\leq \max(1, |\lambda|^{supp_{mn}}) \\ &\quad \times \inf \left\{ t^{q_{mn}/H} : \left( \left[ f_{mn} \left( \left\| \mu_{mn}(\lambda x), (d(x_1), d(x_2), \dots, \right. \right. \right. \right. \right. \right. \right. \\ & \quad \left. \left. \left. \left. \left. d(x_{n-1})) \right\|_p^\varphi \right) \right]^{q_{mn}} \right)^{1/H} \leq 1 \right\}. \end{aligned}$$

□

**Theorem 3.** The  $\beta$ -dual space of  $\left[ \chi_{f\mu}^{2q}, \left\| \mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1})) \right\|_p^\varphi \right]^\beta = \left[ \Lambda_{f\mu}^{2q}, \left\| \mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1})) \right\|_p^\varphi \right]$ .

*Proof.* First, we observe that

$$\begin{aligned} & \left[ \chi_{f\mu}^{2q}, \left\| \mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1})) \right\|_p^\varphi \right]^\beta \\ & \subset \left[ \Gamma_{f\mu}^{2q}, \left\| \mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1})) \right\|_p^\varphi \right]. \end{aligned}$$

Therefore,

$$\begin{aligned} & \left[ \Gamma_{f\mu}^{2q}, \left\| \mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1})) \right\|_p^\varphi \right]^\beta \\ & \subset \left[ \chi_{f\mu}^{2q}, \left\| \mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1})) \right\|_p^\varphi \right]^\beta. \end{aligned}$$

But

$$\left[ \Gamma_{f\mu}^{2q} \right]^\beta \subsetneq \left[ \Lambda_{f\mu}^{2q}, \left\| \mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1})) \right\|_p^\varphi \right].$$

Hence,

$$\begin{aligned} & \left[ \Lambda_{f\mu}^{2q}, \left\| \mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1})) \right\|_p^\varphi \right] \\ & \subset \left[ \chi_{f\mu}^{2q}, \left\| \mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1})) \right\|_p^\varphi \right]^\beta. \end{aligned} \quad (6)$$

Next, we show that

$$\begin{aligned} & \left[ \chi_{f\mu}^{2q}, \left\| \mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1})) \right\|_p^\varphi \right]^\beta \\ & \subset \left[ \Lambda_{f\mu}^{2q}, \left\| \mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1})) \right\|_p^\varphi \right]. \end{aligned}$$

Let  $y = (y_{mn}) \in \left[ \chi_{f\mu}^{2q}, \left\| \mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1})) \right\|_p^\varphi \right]^\beta$ . Consider  $f(x) = \sum_{m=1}^\infty \sum_{n=1}^\infty x_{mn} y_{mn}$  with

$$x = (x_{mn}) \in \left[ \chi_{f\mu}^{2q}, \left\| \mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1})) \right\|_p^\varphi \right]$$

$$x = [(\lambda_{mn} - \lambda_{mn+1}) - (\lambda_{m+1n} - \lambda_{m+1n+1})]$$

$$= \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & & & & & \\ 0 & 0 & \dots & \frac{\varphi_{rs}}{\Delta \lambda_{mn}(m+n)!} & \frac{-\varphi_{rs}}{\Delta \lambda_{mn}(m+n)!} & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$- \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & & & & & \\ 0 & 0 & \dots & \frac{\varphi_{rs}}{\Delta \lambda_{mn}(m+n)!} & \frac{-\varphi_{rs}}{\Delta \lambda_{mn}(m+n)!} & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$\begin{aligned} & \left[ f_{mn} \left( \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right) \right] \\ &= \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0, & \dots & 0 \\ \vdots & & & & & & \\ 0 & 0 & \dots & f_{mn} \left( \frac{\varphi_{rs}}{\Delta \lambda_{mn}(m+n)!} \right) f_{mn} \left( \frac{-\varphi_{rs}}{\Delta \lambda_{mn}(m+n)!} \right) & \dots & 0 \\ 0 & 0 & \dots & f_{mn} \left( \frac{-\varphi_{rs}}{\Delta \lambda_{mn}(m+n)!} \right) f_{mn} \left( \frac{\varphi_{rs}}{\Delta \lambda_{mn}(m+n)!} \right) & \dots & 0 \\ 0 & 0 & \dots & 0 & 0, & \dots & 0 \end{pmatrix}. \end{aligned}$$

Hence, it converges to zero.

Therefore,

$$\begin{aligned} & [(\lambda_{mn} - \lambda_{mn+1}) - (\lambda_{m+1n} - \lambda_{m+1n+1})] \\ & \in \left[ \chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]. \end{aligned}$$

Hence,  $d((\lambda_{mn} - \lambda_{mn+1}) - (\lambda_{m+1n} - \lambda_{m+1n+1}), 0) = 1$ .  
But

$$\begin{aligned} |y_{mn}| &\leq \|f\| d((\lambda_{mn} - \lambda_{mn+1}) - (\lambda_{m+1n} - \lambda_{m+1n+1}), 0) \\ &\leq \|f\| \cdot 1 < \infty \end{aligned}$$

for each  $m, n$ . Thus,  $(y_{mn})$  is a  $p$ - metric paranormed space of double analytic sequence and, hence, an  $p$ - metric double analytic sequence.

In other words,  $y \in \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$ . But  $y = (y_{mn})$  is arbitrary in  $\left[ \chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]^\beta$ . Therefore,

$$\begin{aligned} & \left[ \chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]^\beta \\ & \subset \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]. \end{aligned} \quad (7)$$

From (6) and (7), we get

$$\begin{aligned} & \left[ \chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]^\beta \\ &= \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]. \end{aligned}$$

**Theorem 4.** The dual space of  $\left[ \chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$  is  $\left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$ . In other words,  $\left[ \chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]^* = \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$ .

*Proof.* We recall that

$$\lambda_{mn} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots \\ \vdots & & & & & \\ 0 & 0 & \dots & \frac{\varphi_{rs}}{\Delta \lambda_{mn}(m+n)!} & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots \end{pmatrix}$$

with  $\frac{\varphi_{rs}}{\Delta \lambda_{mn}(m+n)!}$  in the  $(m, n)$ th position and zeros elsewhere,

$$\begin{aligned} & \left[ \chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right] \\ &= \begin{pmatrix} 0. & . & . & 0 \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 0 & f \left( \frac{\varphi_{rs}}{\Delta \lambda_{mn}(m+n)!} \right)^{1/m+n} & . & 0 \\ 0 & (m, n)^{\text{th}} & . & 0 \end{pmatrix} \end{aligned}$$

which is a  $p$ - metric of double gai sequence. Hence,

$$\begin{aligned} x &\in \left[ \chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right] f(x) \\ &= \sum_{m,n=1}^{\infty} x_{mn} y_{mn} \end{aligned}$$

with  $x \in \left[ \chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$  and  $f \in \left[ \chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]^*$ , where  $\left[ \chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]^*$  is the dual space of  $\left[ \chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$ .

Take  $x = (x_{mn}) \in \left[ \chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$ . Then,

$$|y_{mn}| \leq \|f\| d(\varphi_{rs}, 0) < \infty \forall m, n. \quad (8)$$

Thus,  $(y_{mn})$  is a  $p$ - metric of the double analytic sequence and an  $p$ - metric of double analytic sequence.

In other words,  $y \in \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$ . Therefore,

$$\begin{aligned} & \left[ \chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]^* \\ &= \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]. \end{aligned}$$

This completes the proof.  $\square$

**Theorem 5.** (1) If the sequence  $(f_{mn})$  satisfies uniform  $\Delta_2$ -condition, then

$$\begin{aligned} & \left[ \chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]^\alpha \\ &= \left[ \chi_g^{2q\mu}, \|\mu_{uv}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]. \end{aligned}$$

(2) If the sequence  $(g_{mn})$  satisfies uniform  $\Delta_2$ -condition, then

$$\begin{aligned} & \left[ \chi_g^{2q\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]^\alpha \\ &= \left[ \chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]. \end{aligned}$$

*Proof.* Let the sequence  $(f_{mn})$  satisfies uniform  $\Delta_2$ -condition; we get

$$\begin{aligned} & \left[ \chi_g^{2q\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right] \\ & \subset \left[ \chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]^\alpha. \end{aligned} \quad (9)$$

To prove the inclusion

$$\begin{aligned} & \left[ \chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]^\alpha \\ & \subset \left[ \chi_g^{2q\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right], \end{aligned}$$

let  $a \in \left[ \chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]^\alpha$ .

Then, for all  $\{x_{mn}\}$  with  $(x_{mn}) \in \left[ \chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$ , we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn} a_{mn}| < \infty. \quad (10)$$

Since the sequence  $(f_{mn})$  satisfies the uniform  $\Delta_2$ -condition and then

$$(y_{mn}) \in \left[ \chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right],$$

we get  $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left| \frac{\varphi_{rs} y_{mn} a_{mn}}{\Delta \lambda_{mn}(m+n)!} \right| < \infty$ . by (10). Thus,

$$\begin{aligned} (\varphi_{rs} a_{mn}) & \in \left[ \chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right] \\ &= \left[ \chi_g^{2q\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right], \text{ and} \end{aligned}$$

hence,  $(a_{mn}) \in \left[ \chi_g^{2q\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$ . This gives that

$$\begin{aligned} & \left[ \chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]^\alpha \\ & \subset \left[ \chi_g^{2q\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]. \end{aligned} \quad (11)$$

We are granted with (9) and (11) that

$$\begin{aligned} & \left[ \chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]^\alpha \\ &= \left[ \chi_g^{2q\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]. \end{aligned}$$

(3) Similarly, one can prove that

$$\begin{aligned} & \left[ \chi_g^{2q\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]^\alpha \\ & \subset \left[ \chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right] \end{aligned}$$

if the sequence  $(g_{mn})$  satisfies the uniform  $\Delta_2$ -condition.  $\square$

**Proposition 1.** If  $0 < q_{mn} < p_{mn} < \infty$  for each  $m$  and  $n$ , then

$$\begin{aligned} & \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right] \\ & \subseteq \left[ \Lambda_{f\mu}^{2p}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]. \end{aligned}$$

*Proof.* Let  $x = (x_{mn}) \in \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$ . We have

$$\sup_{mn} \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right] < \infty.$$

This implies that

$$\left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right] < 1$$

for sufficiently large value of  $m$  and  $n$ . Since  $f_{mn}$ s are non-decreasing, we get

$$\begin{aligned} & \sup_{mn} \left[ \Lambda_{f\mu}^{2p}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right] \\ & \leq \sup_{mn} \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]. \end{aligned}$$

Thus,  $x = (x_{mn}) \in \left[ \Lambda_{f\mu}^{2p}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$ .  $\square$

**Proposition 2.** (1) If  $0 < \inf q_{mn} \leq q_{mn} < 1$ , then

$$\begin{aligned} & \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right] \\ & \subset \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]. \end{aligned}$$

(2) If  $1 \leq q_{mn} \leq \sup q_{mn} < \infty$ , then  $\left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right] \subset \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$ .

*Proof.* Let  $x = (x_{mn}) \in \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$ . Since  $0 < \inf q_{mn} \leq 1$ , we have

$$\sup_{uv} \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right] \\ \leq \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right],$$

and hence

$$x = (x_{mn}) \in \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right].$$

(3) Let  $q_{mn}$  for each  $(m, n)$  and  $\sup_{mn} q_{mn} < \infty$ .

Let  $x = (x_{mn}) \in \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$ . Then, for each  $0 < \epsilon < 1$ , there exists a positive integer  $\mathbb{N}$  such that

$$\sup_{uv} \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right] \leq \epsilon < 1,$$

for all  $m, n \geq N$ . This implies that

$$\sup_{mn} \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right] \\ \leq \sup_{mn} \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right].$$

Thus,  $x = (x_{mn}) \in \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$ .  $\square$

**Proposition 3.** Let  $f' = (f'_{mn})$  and  $f'' = (f''_{mn})$  be sequences of Musielak functions; we have

$$\left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right] \\ \times \bigcap \left[ \Lambda_{f''\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right] \\ \times \subseteq \left[ \Lambda_{f'+f''\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right].$$

*Proof.* The proof is easy, so we omit it.  $\square$

**Proposition 4.** For any sequence of Musielak functions  $f = (f_{mn})$  and  $q = (q_{mn})$  be double analytic sequence of strictly positive real numbers. Then,

$$\left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right] \\ \subset \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right].$$

*Proof.* The proof is easy, so we omit it.  $\square$

**Proposition 5.** The sequence space  $\left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$  is solid.

*Proof.* Let  $x = (x_{mn}) \in \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$ , i.e.,

$$\sup_{mn} \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right] < \infty.$$

Let  $(\alpha_{mn})$  be double sequence of scalars such that  $|\alpha_{mn}| \leq 1$  for all  $m, n \in N \times N$ . Then, we get

$$\sup_{mn} \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(\alpha x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right] \\ \leq \sup_{mn} \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right].$$

$\square$

**Proposition 6.** The sequence space  $\left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$  is monotone.

*Proof.* The proof follows from Proposition 5.  $\square$

**Proposition 7.** If  $f = (f_{mn})$  is any Musielak function, then

$$\left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi^*} \right] \\ \subset \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi^{**}} \right]$$

if and only if  $\sup_{r,s \geq 1} \frac{\varphi_{rs}^*}{\varphi_{rs}^{**}} < \infty$ .

*Proof.* Let  $x \in \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi^*} \right]$  and  $N = \sup_{r,s \geq 1} \frac{\varphi_{rs}^*}{\varphi_{rs}^{**}} < \infty$ . Then, we get

$$\left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi_{rs}^{**}} \right] \\ = N \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi_{rs}^*} \right] \\ = 0.$$

Thus,  $x \in \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi^{**}} \right]$ . Conversely, suppose that

$$\left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi^*} \right] \\ \subset \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi^{**}} \right]$$

and  $x \in \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi^*} \right]$ .

Then,  $\left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi^*} \right] < \epsilon$  for every  $\epsilon > 0$ . Suppose that  $\sup_{r,s \geq 1} \frac{\varphi_{rs}^*}{\varphi_{rs}^{**}} = \infty$ , then there exists a sequence of members  $(r_{s_{jk}})$  such that  $\lim_{j,k \rightarrow \infty} \frac{\varphi_{jk}^*}{\varphi_{jk}^{**}} = \infty$ . Hence, we have  $\left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi_{rs}^*} \right] = \infty$ . Therefore,  $x \notin \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi^{**}} \right]$ , which is a contradiction.  $\square$



**Proposition 8.** If  $f = (f_{mn})$  is any Musielak function, then

$$\left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right] \\ = \left[ \Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi^*} \right]$$

if and only if  $\sup_{r,s \geq 1} \frac{\varphi_{rs}^*}{\varphi_{rs}} < \infty, \sup_{r,s \geq 1} \frac{\varphi_{rs}^{**}}{\varphi_{rs}^*} > \infty$ .

*Proof.* It is easy to prove, so we omit it.  $\square$

**Proposition 9.** The sequence space  $\left[ \chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$  is not solid.

*Proof.* The result follows from the following example. Consider

$$x = (x_{mn}) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & & & \\ 1 & 1 & \dots & 1 \end{pmatrix} \\ \in \left[ \chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right].$$

Let

$$\alpha_{mn} = \begin{pmatrix} -1^{m+n} & -1^{m+n} & \dots & -1^{m+n} \\ -1^{m+n} & -1^{m+n} & \dots & -1^{m+n} \\ \vdots & & & \\ -1^{m+n} & -1^{m+n} & \dots & -1^{m+n} \end{pmatrix},$$

for all  $m, n \in \mathbb{N}$ . Then,  $\alpha_{mn}x_{mn} \notin \left[ \chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$ . Hence,  $\left[ \chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$  is not solid.  $\square$

**Proposition 10.** The sequence space  $\left[ \chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$  is not monotone.

*Proof.* The proof follows from Proposition 9.  $\square$

### Generalized four-dimensional infinite matrix sequence spaces

Let  $A = (a_{k\ell}^{mn})$  be a four-dimensional infinite matrix of complex numbers. Then, we have  $A(x) = (Ax)_{k\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$  which converges for each  $k, \ell$ .

In this section, we introduce the following sequence spaces:

$$\left[ \chi_{f\mu}^{2qA}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right] \\ = \lim_{mn} \left\{ \left[ f_{mn} \left( \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} = 0 \right\},$$

$$\left[ \Lambda_{f\mu}^{2qA}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right] \\ = \sup_{mn} \left\{ \left[ f_{mn} \left( \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} < \infty \right\}.$$

If we take  $f_{mn}(x) = x$ , we get

$$\left[ \chi_{f\mu}^{2qA}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right] \\ = \lim_{mn} \left\{ \left[ \left( \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} = 0 \right\},$$

$$\left[ \Lambda_{f\mu}^{2qA}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right] \\ = \sup_{mn} \left\{ \left[ \left( \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} < \infty \right\}.$$

If we take  $q = (q_{mn}) = 1$ ,

$$\left[ \chi_{f\mu}^{2A}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right] \\ = \lim_{mn} \left\{ \left[ f_{mn} \left( \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right] = 0 \right\},$$

$$\left[ \Lambda_{f\mu}^{2A}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right] \\ = \sup_{mn} \left\{ \left[ f_{mn} \left( \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right] < \infty \right\}.$$

**Theorem 6.** For a Musielak-modulus function,  $f = (f_{mn})$ . Then, the sequence spaces  $\left[ \chi_{f\mu}^{2qA}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$  and  $\left[ \Lambda_{f\mu}^{2qA}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$  are linear spaces over the set of complex numbers  $\mathbb{C}$ .

*Proof.* It is routine verification. Therefore, the proof is omitted.  $\square$

**Theorem 7.** For any Musielak-modulus function  $f = (f_{mn})$  and a double analytic sequence  $q = (q_{mn})$  of strictly positive real numbers, the space  $\left[ \chi_{f\mu}^{2qA}, \| (d(x_1), d(x_2), \dots, d(x_{n-1})) \|_p^\varphi \right]$  is a topological linear space paranormed by

$$g(x) = \inf \left\{ \left( \left[ f_{mn} \left( \| A_{mn} \mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1})) \|_p^\varphi \right) \right]^{q_{mn}} \right)^{1/H} \leq 1 \right\},$$

where  $H = \max(1, \sup_{mn} q_{mn} < \infty)$ .

*Proof.* Clearly,  $g(x) \geq 0$  for  $x = (x_{mn}) \in \left[ \chi_{f\mu}^{2qA}, \| (d(x_1), d(x_2), \dots, d(x_{n-1})) \|_p^{V_2} \right]$ . Since  $f_{mn}(0) = 0$ , we get  $g(0) = 0$ . Conversely, suppose that  $g(x) = 0$ , then  $\inf \left\{ \left( \left[ f_{mn} \left( \| A_{mn} \mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1})) \|_p^\varphi \right) \right]^{q_{mn}} \right)^{1/H} \leq 1 \right\} = 0$ . Suppose that  $A_{mn} \mu_{mn}(x) \neq 0$  for each  $m, n \in \mathbb{N}$ , then

$$\| A_{mn} \mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1})) \|_p^\varphi \rightarrow \infty. \quad (12)$$

It follows that  $\left( \left[ f_{mn} \left( \| A_{mn} \mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1})) \|_p^{V_2} \right) \right]^{q_{mn}} \right)^{1/H} \rightarrow \infty$  which is a contradiction.

Therefore,  $A_{mn} \mu_{mn}(x) = 0$ . Let  $\left( \left[ f_{mn} \left( \| A_{mn} \mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1})) \|_p^\varphi \right) \right]^{q_{mn}} \right)^{1/H} \leq 1$  and  $\left( \left[ f_{mn} \left( \| A_{mn} \mu_{mn}(y), (d(x_1), d(x_2), \dots, d(x_{n-1})) \|_p^\varphi \right) \right]^{q_{mn}} \right)^{1/H} \leq 1$ .

Then, by using Minkowski's inequality, we have

$$\begin{aligned} & \left( \left[ f_{mn} \left( \| A_{mn} \mu_{mn}(x+y), (d(x_1), d(x_2), \dots, d(x_{n-1})) \|_p^\varphi \right) \right]^{q_{mn}} \right)^{1/H} \\ & \leq \left( \left[ f_{mn} \left( \| A_{mn} \mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1})) \|_p^\varphi \right) \right]^{q_{mn}} \right)^{1/H} \\ & \quad + \left( \left[ f_{mn} \left( \| A_{mn} \mu_{mn}(y), (d(x_1), d(x_2), \dots, d(x_{n-1})) \|_p^\varphi \right) \right]^{q_{mn}} \right)^{1/H}. \end{aligned}$$

So, we have

$$\begin{aligned} g(x+y) &= \inf \left\{ \left( \left[ f_{mn} \left( \| A_{mn} \mu_{mn}(x+y), (d(x_1), d(x_2), \dots, d(x_{n-1})) \|_p^\varphi \right) \right]^{q_{mn}} \right)^{1/H} \leq 1 \right\} \\ &\leq \inf \left\{ \left( \left[ f_{mn} \left( \| A_{mn} \mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1})) \|_p^\varphi \right) \right]^{q_{mn}} \right)^{1/H} \leq 1 \right\} \\ &\quad + \inf \left\{ \left( \left[ f_{mn} \left( \| A_{mn} \mu_{mn}(y), (d(x_1), d(x_2), \dots, d(x_{n-1})) \|_p^\varphi \right) \right]^{q_{mn}} \right)^{1/H} \leq 1 \right\}. \end{aligned}$$

Therefore,

$$g(x+y) \leq g(x) + g(y).$$

Finally, to prove that the scalar multiplication is continuous, let  $\lambda$  be any complex number. By definition,

$$g(\lambda x) = \inf \left\{ \left( \left[ f_{mn} \left( \| A_{mn} \mu_{mn}(\lambda x), (d(x_1), d(x_2), \dots, d(x_{n-1})) \|_p^\varphi \right) \right]^{q_{mn}} \right)^{1/H} \leq 1 \right\}.$$

Then,

$$g(\lambda x) = \inf \left\{ (|\lambda| t)^{q_{mn}/H} : \left( \left[ f_{mn} \left( \| A_{mn} \mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1})) \|_p^\varphi \right) \right]^{q_{mn}} \right)^{1/H} \leq 1 \right\},$$

where  $t = \frac{1}{|\lambda|}$ . Since  $|\lambda|^{q_{mn}} \leq \max(1, |\lambda|^{\sup_{mn} q_{mn}})$ , we have

$$g(\lambda x) \leq \max(1, |\lambda|^{\sup_{mn} q_{mn}}) \inf \left\{ t^{q_{mn}/H} : \left( \left[ f_{mn} \left( \| A_{mn} \mu_{mn}(\lambda x), (d(x_1), d(x_2), \dots, d(x_{n-1})) \|_p^\varphi \right) \right]^{q_{mn}} \right)^{1/H} \leq 1 \right\}.$$

□

**Theorem 8.** The  $\beta$ -dual space of  $\left[ \chi_{f\mu}^{2qA}, \| A_{mn} \mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1})) \|_p^\varphi \right]^\beta = \left[ \chi_{f\mu}^{2qA}, \| A_{mn} \mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1})) \|_p^\varphi \right]$ .

*Proof.* First, we observe that

$$\left[ \Gamma_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]^\beta \\ \subset \left[ \Gamma_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right].$$

Therefore,

$$\left[ \Gamma_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]^\beta \\ \subset \left[ \chi_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]^\beta.$$

But

$$\left[ \Gamma_{f\mu}^{2qA} \right]^\beta \subsetneq \left[ \Lambda_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right].$$

Hence,

$$\left[ \Lambda_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right] \\ \subset \left[ \chi_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]^\beta.$$

Next, we show that

$$\left[ \chi_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]^\beta \\ \subset \left[ \Lambda_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right].$$

Let  $y = (y_{mn}) \in \left[ \chi_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]^\beta$ . Consider  $f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x_{mn} y_{mn}$

with

$$x = (x_{mn}) \in \left[ \chi_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$$

$$x = (\lambda_{mn} - \lambda_{mn+1}) - (\lambda_{m+1n} - \lambda_{m+1n+1})$$

$$= \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{\varphi_{rs}}{a_{k\ell}^{mn} \Delta \lambda_{mn}(m+n)!} & \frac{-\varphi_{rs}}{a_{k\ell}^{mn} \Delta \lambda_{mn}(m+n)!} & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$- \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{\varphi_{rs}}{a_{k\ell}^{mn} \Delta \lambda_{mn}(m+n)!} & \frac{-\varphi_{rs}}{a_{k\ell}^{mn} \Delta \lambda_{mn}(m+n)!} & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$\left[ f_{mn} \left( \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right) \right]$$

$$= \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & f_{mn} \left( \frac{\varphi_{rs}}{a_{k\ell}^{mn} \Delta \lambda_{mn}(m+n)!} \right) & f_{mn} \left( \frac{-\varphi_{rs}}{a_{k\ell}^{mn} \Delta \lambda_{mn}(m+n)!} \right) & \dots & 0 \\ 0 & 0 & \dots & f_{mn} \left( \frac{-\varphi_{rs}}{a_{k\ell}^{mn} \Delta \lambda_{mn}(m+n)!} \right) & f_{mn} \left( \frac{\varphi_{rs}}{a_{k\ell}^{mn} \Delta \lambda_{mn}(m+n)!} \right) & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Hence, converges to zero.

Therefore,

$$[(\lambda_{mn} - \lambda_{mn+1}) - (\lambda_{m+1n} - \lambda_{m+1n+1})] \\ \times \in \left[ \chi_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right].$$

Hence,  $d(a_{k\ell}^{mn}(\lambda_{mn} - \lambda_{mn+1}) - (\lambda_{m+1n} - \lambda_{m+1n+1}), 0) = 1$ . However,  $|y_{mn}| \leq \|f\| \cdot d(a_{k\ell}^{mn}(\lambda_{mn} - \lambda_{mn+1}) - (\lambda_{m+1n} - \lambda_{m+1n+1}), 0) \leq \|f\| \cdot 1 < \infty$  for each  $m, n$ . Thus,  $(y_{mn})$  is a  $p$ - metric paranormed space of double analytic sequence and, hence, an  $p$ - metric double analytic sequence.

In other words,  $y \in \left[ \Lambda_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$ . However,  $y = (y_{mn})$  is arbitrary in  $\left[ \chi_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]^\beta$ . Therefore,

$$\left[ \chi_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]^\beta \\ \subset \left[ \Lambda_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]. \quad (13)$$

From (12) and (13), we get

$$\begin{aligned} & \left[ \chi_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]^\beta \\ &= \left[ \Lambda_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]. \end{aligned}$$

□

**Theorem 9.** The dual space of  $\left[ \chi_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$  is  $\left[ \Lambda_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$ . In other words,

$$\begin{aligned} & \left[ \chi_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]^* \\ &= \left[ \Lambda_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]. \end{aligned}$$

*Proof.* We recall that

$$\lambda_{mn} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots \\ \vdots & & & & & \\ 0 & 0 & \dots & \frac{\varphi_{rs}}{a_{k\ell}^{mn} \Delta \lambda_{mn}(m+n)!} & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots \end{pmatrix}$$

with  $\frac{\varphi_{rs}}{a_{k\ell}^{mn} \Delta \lambda_{mn}(m+n)!}$  in the  $(m, n)$ th position and zero elsewhere,

$$\left[ \chi_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$$

$$= \begin{pmatrix} 0. & . & . & 0 \\ \vdots & & & \\ 0 & f\left(\frac{\varphi_{rs}}{a_{k\ell}^{mn} \Delta \lambda_{mn}(m+n)!}\right)^{1/m+n} & . & 0 \\ 0 & (m, n)^{\text{th}} & . & 0 \end{pmatrix}$$

which is a  $p$ - metric of double gai sequence. Hence,

$$\begin{aligned} x &\in \left[ \chi_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right] f(x) \\ &= \sum_{m,n=1}^{\infty} x_{mn} y_{mn} \end{aligned}$$

with

$$x \in \left[ \chi_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$$

and

$$f \in \left[ \chi_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]^*,$$

where

$$\left[ \chi_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]^*$$

is the dual space of  $\left[ \chi_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$ .

Take  $x = (x_{mn}) \in \left[ \chi_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$ . Then,

$$|y_{mn}| \leq \|f\| d(\varphi_{rs}, 0) < \infty \forall m, n. \quad (14)$$

Thus,  $(y_{mn})$  is a  $p$ - metric of double analytic sequence and, hence, an  $p$ - metric of double analytic sequence. In other words,  $y \in \left[ \Lambda_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]$ . Therefore,

$$\begin{aligned} & \left[ \chi_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]^* \\ &= \left[ \Lambda_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]. \end{aligned}$$

□

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors contributed equally in introducing gai-2 sequence spaces generalized over  $p$ -metric defined by Musielak modulus function and in studying topological properties. All authors read and approved the final manuscript.

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